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An analytic description of converging shock waves

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The similarity solution describing the motion of converging spherical and cylindrical shocks is governed by a set of three ordinary differential equations. Previous descriptions of the shock motion have been based on numerical solutions of these differential equations. In the present paper a study of the singular points of the differential equations leads to an analytic description of the flow and a determination of the similarity exponent which is in excellent agreement with the earlier numerical values. Limiting values of the ratio of specific heats are considered. It is shown that as the ratio tends to unity the shock becomes 'freely propagating' and the first terms in a power series for the similarity exponent are obtained. Large values of the ratio of specific heats are briefly considered and provide a further check on the analytic description of this paper. Finally in the Appendix the condition for the pressure to have a maximum is clarified and the location of the maximum provides further strong evidence of the high accuracy of the analytic approach of this paper.

1. Introduction

This paper considers the collapse of cylindrical and spherical shock waves moving through an ideal gas with a constant ratio of specific heats. The medium is initially uniform and at rest. In the final stages of the collapse the shock becomes strong and the pressure ahead of the shock is neglected in comparison with the pressure behind the shock, leading to a similarity formulation of the problem. In this problem the similarity variable, which is the ratio of a distance to a particular power of the time, is not known *a priori*. The particular power, known as the similarity exponent, is determined from the solution of a single ordinary differential equation. The determination of the similarity exponent has been described by many authors, e.g. Guderly (1942), Butler (1954), Sedov (1959), Stanyukovich (1960) and Zel'dovich & Raiser (1967), but there is scant description of the flow behind the shock wave in the literature. This determination of the similarity exponent is outlined in §2.

The main aim of this paper is to provide a simple analytic description of the flow behind collapsing shock waves. This is achieved by replacing the previous approach of numerical solution of ordinary differential equations by a theoretical study of their singular points. This theoretical approach leads, in §3, to an approximate determination of the similarity exponent which is in agreement with the previously obtained values correct to four significant figures over the physically relevant range of ratios of specific heats. These extremely good, though approximate, values of the similarity exponent lead, in §4, to a simple analytic description of the flow variables at all points behind the converging shocks. Two particular comparisons can be made with the numerical solutions. The first is provided by an integral of the differential equations, known as the entropy integral. It is shown that the approximate analytic solution satisfies the entropy integral exactly. The second comparison is provided by the location of the pressure maximum behind the shock, which exists when the ratio of specific heats is somewhat less than 3. In the Appendix, the location of the maximum for the numerical solutions is determined and is found to coincide with the location of the pressure maximum of the analytic solution presented in this paper.

In an earlier paper (Chisnell 1957) the author presented a ‘freely propagating’ description of collapsing cylindrical and spherical shock waves. In that paper the motion of the shock was assumed to be unaffected by disturbances in the flow behind the shock which overtake and modify the motion of the shock. The work was based on that of Chester (1954) on the motion of a shock in a channel containing a small area change. A naive integration of Chester’s result produced a description of converging shock waves which ignored the effect of the overtaking wave on the shock. The 1957 paper gave an explicit approximate formula for the similarity exponent in terms of the ratio of specific heats of the gas. This result was incorporated by Whitham (1957, 1959) in his theory of shock wave dynamics and this approach is often referred to as the CCW method in the literature.

In §5 the motion is considered as the ratio of specific heats tends to unity. It is shown that in this limit the region of disturbance overtaking the shock during its passage to the origin shrinks to zero so that the shock becomes freely propagating. An alternative derivation of the 1957 freely propagating law appears as the lowest order approximation. Finally in §6 the motion is considered for large values of the ratio of specific heats to provide a further check on the analytic solution of this paper.

2. The similarity formulation

The equations of motion for symmetric adiabatic flow of an ideal gas are

$$\rho_t + u\rho_r + \rho r^{1-s}(ur^{s-1})_r = 0, \tag{2.1a}$$

$$u_t + uu_r + \frac{1}{\rho}p_r = 0, \tag{2.1b}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)(\ln(p/\rho^\gamma)) = 0, \tag{2.1c}$$

where r is the distance from the origin, O, and the space index s is 2 for cylindrical flow and 3 for spherical flow. The pressure p is replaced by the speed of sound c, where $c^2 = \gamma p/\rho$, and then non-dimensional variables G, V, Z are introduced with

$$\rho = \rho_0 G, \qquad u = \frac{r}{t}V, \qquad c^2 = \frac{r^2}{t^2}Z. \tag{2.2a,b,c}$$

The motion of the shock takes place during negative time, arriving at the origin O at time $t = 0$. The equations for G, V, Z are

$$tG_t + VrG_r + GrV_r = -sVG, \tag{2.3a}$$

$$tV_t + VrV_r + \frac{1}{\gamma}\frac{Z}{G}rG_r + \frac{1}{\gamma}rZ_r = V - V^2 - \frac{2Z}{\gamma}, \tag{2.3b}$$

$$t\frac{Z_t}{Z} + rV\frac{Z_r}{Z} - \frac{\gamma-1}{G}(tG_t + VrG_r) = 2 - 2V. \tag{2.3c}$$

Self-similar solutions of these equations are sought in terms of the variable $\xi = r/R$, where $R(t)$ is the distance of the shock from the origin at time $t(<0)$ and G, V, Z are functions of ξ alone. Changing the independent variables from r, t to ξ, t using

$$\frac{\partial}{\partial r} = \frac{1}{R}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \xi\frac{\dot{R}}{R}\frac{\partial}{\partial \xi}, \tag{2.4}$$

where $\dot{R}$ denotes $\mathrm{d}R/\mathrm{d}t$, it is noted that the t-variable enters the equations only in the combination $t\dot{R}/R$ arising from the $t\partial/\partial t$ terms in (2.3). Thus for self-similar solutions to exist

$$\frac{t\dot{R}}{R} = \alpha \qquad \text{or} \qquad R = A(-t)^{\alpha}, \tag{2.5}$$

where α and A are constants. The resulting equations for $V(\xi)$, $G(\xi)$, $Z(\xi)$ are

$$\xi V' + (V-\alpha)\xi\frac{G'}{G} = -sV, \tag{2.6a}$$

$$(V-\alpha)\xi V' + \frac{Z}{\gamma}\xi\frac{G'}{G} + \frac{1}{\gamma}\xi Z' = V - V^2 - \frac{2Z}{\gamma}, \tag{2.6b}$$

$$(\gamma-1)Z\xi\frac{G'}{G} - \xi Z' = -\frac{2Z(1-V)}{V-\alpha}. \tag{2.6c}$$

These equations have a determinantal solution

$$\xi V' = \frac{\Delta_1}{\Delta}, \qquad \xi\frac{G'}{G} = \frac{\Delta_2}{\Delta}, \qquad \xi Z' = \frac{\Delta_3}{\Delta}, \tag{2.7a,b,c}$$

where

$$\Delta = -Z + (V-\alpha)^2 \tag{2.8}$$

is the determinant of the left-hand-side coefficients and Δ_i is the determinant obtained by replacing the ith column of Δ by the right-hand side of (2.6). The three determinants Δ_i can each be expressed in terms of Δ and a particular V-quadratic

$$Q(V) = sV(V-\alpha) + \frac{2(1-\alpha)}{\gamma}(\alpha - V) - V(V-1) \tag{2.9}$$

as follows:

$$\Delta_1 = -\Delta\left\{sV - \frac{2(1-\alpha)}{\gamma}\right\} - (\alpha - V)Q(V), \tag{2.10a}$$

$$\Delta_2 = \frac{2(1-\alpha)}{\gamma(\alpha - V)}\Delta - Q(V), \tag{2.10b}$$

$$\Delta_3 = \frac{Z}{V-\alpha}\left[2\Delta\left\{\alpha - V + \frac{1-\alpha}{\gamma}\right\} + (\gamma-1)(\alpha-V)Q(V)\right]. \tag{2.10c}$$

As ξ occurs in (2.7) solely as the coefficient of the ξ-derivatives and G does not occur in the determinants, the system of equations may be decoupled to provide a single ordinary differential equation

$$\frac{\mathrm{d}Z}{\mathrm{d}V} = \frac{\Delta_3}{\Delta_1} \tag{2.11}$$

and two supplementary equations

$$\frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}V} = \frac{\Delta_2}{\Delta_1}, \qquad \frac{1}{\xi}\frac{\mathrm{d}\xi}{\mathrm{d}V} = \frac{\Delta}{\Delta_1}, \tag{2.12a,b}$$

which can be solved subsequently when $Z(V)$ has been determined from (2.11).

To describe the flow behind the shock, a solution of the equations (2.7) is required which, at the shock $\xi = 1$, satisfies the Rankine–Hugoniot conditions for a strong shock

$$\frac{\rho}{\rho_0} = \frac{\gamma+1}{\gamma-1}, \qquad u = \frac{2}{\gamma+1}\dot{R}, \qquad c^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\dot{R}^2. \tag{2.13a,b,c}$$

To incorporate these conditions into the present paper we express the flow variables of (2.2) in terms of ξ, t and use (2.5) to remove R/t, providing

$$\rho = \rho_0 G, \quad u = \frac{\dot{R}}{\alpha}\xi V, \quad p = \frac{\rho_0 \dot{R}^2}{\gamma\alpha^2}\xi^2 GZ, \quad c^2 = \frac{\dot{R}^2}{\alpha^2}\xi^2 Z. \tag{2.14a,b,c,d}$$

The shock boundary conditions become

$$\xi = 1, \quad G_s = \frac{\gamma+1}{\gamma-1}, \quad V_s = \frac{2\alpha}{\gamma+1}, \quad Z_s = \frac{2\gamma(\gamma-1)\alpha^2}{(\gamma+1)^2}, \tag{2.15a,b,c,d}$$

the s-suffix denoting conditions at the shock. Far behind the shock, where r is large, (2.2) shows that

$$V(\infty) = 0, \qquad Z(\infty) = 0. \tag{2.16a,b}$$

In the (V, Z)-plane, $\Delta = 0$ is a parabola touching the V-axis at $V = \alpha$. Equations (2.15) show that Δ has the negative value $-\alpha^2(\gamma-1)/(\gamma+1)$ at the shock $\xi = 1$. Thus the solution curve in the (V, Z)-plane has to cross the parabola in order to reach the origin, which represents conditions far behind the shock. The solution curve and the parabola $\Delta = 0$ are shown in figure 1 for spherical shocks with $\gamma = 1.4$. Equations (2.7) show that at a point on the parabola $\Delta = 0$ the flow variables will have infinite slopes unless Δ_i simultaneously vanish. Equations (2.10) show that this must occur at a value of V which is a zero of $Q(V)$. For a given value of γ there is a value of the similarity exponent α which provides a solution of (2.11) which starts at the singular point on $\Delta = 0$ and passes through the shock point (2.15). The nature of the singular point is determined at the end of the next section. The iteration required to find α involves first a choice of one of the two zeros of $Q(V)$ and then choosing the positive value of the two possible slopes of the solution curve at the singular point. The equation may then be integrated from the singular point back to the origin, which is a nodal point attracting all solution curves, and the solution is completed by integrating equations (2.12). This description of the numerical determination of the similarity exponent is based on the formulation of Zel'dovich & Raiser (1967).

3. Determination of the similarity exponent

In this section the behaviour of the differential equation which determines the similarity exponent is examined at two of its singular points. The equation given in (2.11) may be written

$$\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}V} = \frac{2\Delta(\alpha - V + (1-\alpha)/\gamma) + (\gamma-1)(\alpha-V)Q}{\Delta(sV - 2(1-\alpha)/\gamma)(\alpha-V) + (\alpha-V)^2 Q}, \tag{3.1}$$

after using (2.10). By taking note of the correct behaviour of Z at these singular points, a trial function, Z_T, is constructed which when substituted into Δ in the right-hand side of (3.1) leads to extremely accurate values of the similarity exponent α and a simple analytic description of the flow. The iteration required to determine the similarity exponent α is performed after the integration, rather than before the

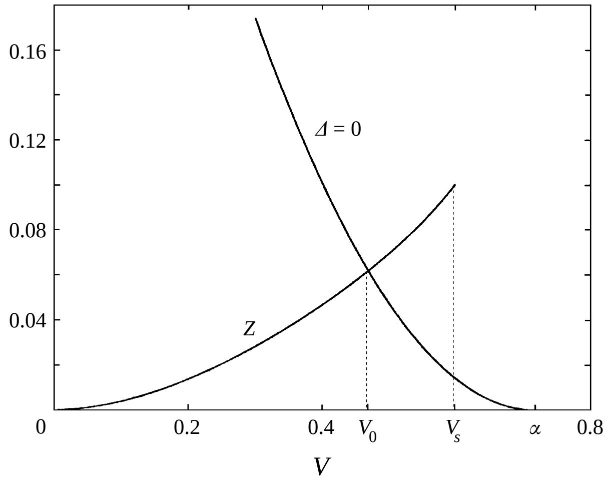


FIGURE 1. The solution curve Z and the parabola $\Delta = 0$ are shown for spherical shocks with $\gamma = 1.4$. The solution curve has a singular point at V_0, the shock is at V_s and the origin corresponds to conditions far behind the shock.

integration as in the earlier numerical similarity solutions. It will be shown that the use of the trial function Z_T in Δ enables the singular behaviour of the solution as it crosses the parabola $\Delta = 0$ to be removed, even though the location of the singular point is not known when the integration is performed.

The first singular point considered is the origin in the (V, Z)-plane which is at one end of the range of integration and corresponds to conditions far behind the shock. Expanding Δ and Q given in (2.8) and (2.9) as far as the linear terms provides

$$\Delta \sim \alpha^2 - 2\alpha V - Z, \qquad Q \sim 2\alpha(1-\alpha)/\gamma - V(s\alpha - 1 + 2(1-\alpha)/\gamma).$$

Substsitution in (3.1) provides

$$\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}V} \sim \frac{2\alpha^2}{V\alpha^2 + 2Z\alpha(1-\alpha)/\gamma}$$

which has the solution

$$V = \frac{2(1-\alpha)}{\gamma\alpha} Z + A_1 Z^{1/2}. \tag{3.2}$$

Provided the integration constant A_1 is non-zero the leading term for Z in terms of V is

$$Z = A_2 V^2. \tag{3.3}$$

The second singular point to be considered arises from the factor $(\alpha - V)$ in the denominator of (3.1). The boundary condition $V_s = 2\alpha/(\gamma + 1)$ given in (2.15*c*) shows that the singular point $V = \alpha$ lies outside the range of integration. Near $V = \alpha$ the limiting form of the equation is

$$\frac{\mathrm{d}Z}{\mathrm{d}V} \sim \frac{Z\ 2(1-\alpha)/\gamma}{(\alpha - V)(s\alpha - 2(1-\alpha)/\gamma)},$$

showing that

$$Z \sim A_3(\alpha - V)^{-\eta}, \qquad \eta^{-1} = \frac{s\alpha\gamma}{2(1-\alpha)} - 1. \tag{3.4a,b}$$

Removal of the singular behaviour at $V = \alpha$ leads to the differential equation

$$\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}V} - \frac{\eta}{\alpha - V} = \frac{\varDelta(2 + s\eta) + (\gamma - 1 - \eta)Q}{\varDelta(sV - 2(1-\alpha)/\gamma) + (\alpha - V)Q} \tag{3.5}$$

to be used in place of (3.1). Any trial function $Z_T(V)$ which is inserted in $\varDelta$ in the right-hand side of (3.5) and is finite at $V = \alpha$ will result in the Z obtained by integrating (3.5) having the correct behaviour as $V \to \alpha$. The integration of (3.5) has to be performed without knowing, *a priori*, the value of V at which the solution crosses the parabola $\varDelta = 0$. If this value is $V = V_0$ then $Q(V_0) = 0$ and $Z_T(V_0)$ must equal $(\alpha - V_0)^2$. The trial function

$$Z_T = \lambda V^2, \qquad \lambda = (\alpha/V_0 - 1)^2, \tag{3.6}$$

has the required value at V_0, is finite at $V = \alpha$ and has the behaviour given by (3.3) for small V.

This trial function is seen from (2.2*b,c*) to correspond to constant-Mach-number flow behind the shock. Insertion of the trial function into $\varDelta$ in the right-hand side of (3.5) enables the singular behaviour at $V = V_0$ to be removed as both $\varDelta$ and Q will have the factor $(V - V_0)$. As V_0 cannot be determined until after the integration has been performed, the factorization must be performed with V_0 as a parameter. Substitution of (3.6) into the definitions (2.8), (2.9) gives the required forms

$$\varDelta = (V - V_0)(V(1-\lambda) - \alpha^2/V_0), \tag{3.7a}$$

$$Q = (V - V_0)\left((s-1)V - \frac{2\alpha(1-\alpha)}{\gamma V_0}\right). \tag{3.7b}$$

Substitution into the right-hand side of (3.5) and removing the $(V - V_0)$ factor leaves a quotient with a quadratic denominator having a zero constant term

$$\frac{V\{\eta(1 - s\lambda) + 2(1-\lambda) + (s-1)(\gamma-1)\} - 2\alpha/V_0}{(1 - s\lambda)V^2 - \alpha V/V_0}. \tag{3.8}$$

The given forms of the constant term in the numerator and the V-coefficient in the denominator follow after using the η-definition (3.4*b*) and (2.9) to express $Q(V_0) = 0$. After expressing (3.8) in partial fractions equation (3.5) becomes

$$\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}V} - \frac{\eta}{\alpha - V} = \frac{2}{V} + \frac{B}{V + q}, \tag{3.9a}$$

$$B = \eta + (s-1)\frac{2\lambda + \gamma - 1}{1 - s\lambda}, \tag{3.9b}$$

$$q = \frac{-\alpha}{V_0(1 - s\lambda)}. \tag{3.9c}$$

The integration of (3.9*a*) is displayed in figure 1 for spherical shocks with $\gamma = 1.4$. A solution of this equation is required which passes through the singular point V_0, Z_0 and the shock point V_s, Z_s. Integration from V_0 to V_s gives

$$\frac{Z_s}{Z_0} = \left(\frac{\alpha - V_0}{\alpha - V_s}\right)^{\eta}\left(\frac{V_s}{V_0}\right)^2\left(\frac{V_s + q}{V_0 + q}\right)^B$$

as the condition to be satisfied by the parameters α, V_0. Substituting V_s, Z_s from (2.15*c,d*) and $Z_0 = (\alpha - V_0)^2$ provides

$$(\alpha/V_0 - 1)^2 = \frac{\gamma(\gamma-1)}{2}\left(\frac{\gamma - 1}{(\gamma+1)(1 - V_0/\alpha)}\right)^{\eta}\left(\frac{V_0 + q}{2\alpha/(\gamma+1) + q}\right)^B. \tag{3.10}$$

(*a*) γ	This paper α/V_0	This paper $\alpha/(1-\alpha)$	Exact $\alpha/(1-\alpha)$
1.2	1.33137	**3.118**38	3.11769
1.4	1.53064	**2.535**58	2.53573
5/3	1.78035	**2.208**95	2.20902
2	2.08607	**2.00341**	2.00341
3	3.00187	**1.750**39	1.75036

(*b*) γ	This paper α/V_0	This paper $\alpha/(1-\alpha)$	Exact $\alpha/(1-\alpha)$
1.2	1.33594	**6.203**29	6.20269
1.4	1.53008	**5.072**35	5.07249
5/3	1.76805	**4.4236**6	4.42373
2	2.05445	**4.0028**1	4.00280
3	2.89436	**3.457**46	3.45766

TABLE 1. The values of $\alpha/(1-\alpha)$ obtained from equations (3.10) and (3.11) for (*a*) spherical shocks and (*b*) cylindrical shocks. The values are compared with the exact values of the parameter obtained by integration of the full equation (3.1). The significant figures in agreement with the exact values are shown in bold and it is seen that there is agreement to at least four significant figures. The values of α/V_0 from equation (3.10) are also given so that the constants in the solution may be calculated.

The parameters α, V_0 are also related by the condition $Q(V_0) = 0$ with Q given in (2.9). By rewriting the last bracket $(V-1)$ in (2.9) as $((V-\alpha)+(\alpha-1))$ the quadratic determination of V_0 in terms of α can be replaced by

$$\frac{(s-1)\alpha}{1-\alpha} = \frac{1}{1-V_0/\alpha} + \frac{2}{\gamma V_0/\alpha}. \tag{3.11}$$

This explicit determination of $\alpha/(1-\alpha)$ in terms of V_0/α suggests that V_0/α be used as the iteration parameter in the solution of (3.10). For a given V_0/α, having determined $\alpha/(1-\alpha)$, the parameter α and hence V_0 follow and may be inserted in the right-hand side of (3.10). Putting $\eta = 0$ and $B = 0$ in (3.10) provides a good first approximation to α/V_0 and this choice of α/V_0 is discussed in §5. For $\gamma = 1.4$ six-figure accuracy for α/V_0 is obtained after just one iteration and the slowest convergence over the range of γ given in table 1 occurs at $\gamma = 3$ where five iterations are needed for five-figure accuracy. The results for α/V_0 and the similarity exponent $\alpha/(1-\alpha)$ are given in table 1 and the similarity exponent is compared with the exact values obtained by numerical integration of the full equation (3.1) for both spherical and cylindrical shocks.

When γ approaches unity, α/V_0 also approaches unity and $\alpha/(1-\alpha)$ becomes large. Hence α and V_0 tend to unity while the second zero of $Q(V)$ is seen from (3.7*b*) to become small. This limit is discussed in §5. As γ increases from unity, the values of $\alpha/(1-\alpha)$ and α decrease and the zeros of $Q(V)$ move closer together. By expressing $Q(V)$ given in (2.9) in the form

$$\frac{Q(V)}{(s-1)\alpha^2} = \left(\frac{V}{\alpha} - \frac{1}{2}\left(1 + \frac{1}{\beta}\left(\frac{2}{\gamma} - 1\right)\right)\right)^2 - \frac{1}{4\beta^2}\left(\left(\beta - \left(\frac{2}{\gamma}+1\right)\right)^2 - \frac{8}{\gamma}\right) \tag{3.12}$$

where

$$\beta = (s-1)\alpha/(1-\alpha), \tag{3.13}$$

it is seen that $Q(V)$ has repeated zeros when β has decreased to the value

$$\beta = 2/\gamma + 1 + (8/\gamma)^{1/2} = (1 + (2/\gamma)^{1/2})^2.$$

This particular location of the singular point V_0 may be expressed as

$$\alpha/V_0 = 1 + (\gamma/2)^{1/2}. \tag{3.14}$$

For the approximate solution of this section the zeros become coincident at $\gamma = 1.870$ for spherical shocks and at $\gamma = 1.909$ for cylindrical shocks. These four-figure values agree with the corresponding values obtained by Lazarus (1981) for the full numerical solution. As γ increases further the zeros of $Q(V)$ move apart again and the singular point V_0 becomes the smaller of the two zeros. It will be shown in the next section that the larger zero moves outside the range $(0, V_s)$ before γ has reached 3. When γ becomes large, the singular point V_0 and the shock point V_s both become small. This limit is discussed in §6.

The transition of the singular point V_0 from being the larger to the smaller zero of $Q(V)$ produces a change in the nature of the singular point behaviour. In the literature the singular point is usually referred to as a saddle point, which is true for $\gamma \leqslant 5/3$ when V_0 is the larger zero. However, when V_0 becomes the smaller zero, which occurs for $\gamma \geqslant 2$, the singular point has nodal behaviour. To demonstrate this result, substitute $Z = Z_0 + (\alpha - V_0)z$, $V = V_0 + v$ into equation (3.1) and linearize, yielding

$$\frac{\mathrm{d}z}{\mathrm{d}v} = \frac{v(4e - (\gamma - 1)g) + 2ez}{v(2f - g) + fz}, \tag{3.15}$$

where

$$e = (\alpha - V_0) + (1 - \alpha)/\gamma, \qquad f = sV_0 - 2(1 - \alpha)/\gamma, \tag{3.16a,b}$$

$$g = (s - 1)V_0 - 2\alpha(1 - \alpha)/(\gamma V_0). \tag{3.16c}$$

The coefficients e, f remain positive for all γ, and g is seen from (3.7b) to be positive only when V_0 is the larger zero. The solution of (3.15) is

$$(z - \phi_1 v)^{\phi_1 + 2 - g/f} = C(z - \phi_2 v)^{\phi_2 + 2 - g/f}, \tag{3.17}$$

where ϕ_1, ϕ_2 are the roots of the quadratic

$$f\phi^2 - (2e - 2f + g)\phi - 4e + (\gamma - 1)g = 0 \tag{3.18}$$

and C is a constant. When $Q(V)$ has repeated zeros, $g = 0$ and the roots of (3.18) are $\phi_1 = 2e/f$ and $\phi_2 = -2$. When V_0 is the larger zero, g is positive so that the exponents in (3.17) are of opposite sign producing saddle point behaviour. The required solution curve with positive slope is $z = \phi_1 v$. When V_0 is the smaller zero, the exponents have the same sign giving nodal behaviour with many solution curves that are tangential to $z = \phi_2 v$ with negative slope. The remaining solution curve, which corresponds to $C = 0$, is the required solution curve with positive slope, $z = \phi_1 v$. Thus in both cases the solution curve $Z(V)$ has slope $(\alpha - V_0)\phi_1$ at the singular point.

This exact value for the slope at the singular points provides another comparison with the approximate theory of this section. Using the trial function Z_T, given in (3.6), in the right-hand side of (3.1) the slope at V_0 follows as

$$(\alpha - V_0)\frac{4e\alpha/V_0 - (\gamma - 1)g}{2f\alpha/V_0 - g}.$$

When $Q(V)$ has repeated zeros, both theories give the same expression for the slope $2(\alpha - V_0)e/f$ and over the physical range of γ the slopes are in agreement to three significant figures for $\gamma \leqslant 2$ and to two figures at $\gamma = 3$.

4. The flow field

In the previous section, the similarity exponent α was determined for various γ by finding a solution of the V, Z differential equation which passed through the singular point $V_0, Z_0 = (\alpha - V_0)^2$ and the shock point V_s, Z_s. Having determined the correct values of α and V_0, the supplementary differential equations for G and ξ in (2.12) are now integrated. These equations also have a singular point at V_0 and use of the trial function Z_T given in (3.6) enables the factor $(V - V_0)$ to be removed from the numerators and denominators of the right-hand sides of (2.12). Appropriate expansions for $\varDelta$ and Q are given in (3.7*a,b*) and the corresponding expansions for $\varDelta_1$ and $\varDelta_2$, defined in (2.10*a,b*) are

$$\varDelta_1 = -(V - V_0)V(1 - s\lambda)(V + q),$$
$$\varDelta_2 = (V - V_0)V\{(s-1)(V-\alpha) + 2(1-\alpha)\alpha(1-\alpha/V_0)/(\gamma V_0)\}/(\alpha - V).$$

Substitution into (2.12) provides

$$\frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}V} = \frac{E}{V-\alpha} + \frac{D}{V+q}, \qquad \frac{1}{\xi}\frac{\mathrm{d}\xi}{\mathrm{d}V} = -\frac{\alpha}{V} + \frac{F}{V+q}, \tag{4.1a,b}$$

where

$$E = \frac{2(1-\alpha)(1-\alpha/V_0)/\gamma}{(1-s\lambda)V_0 - 1}, \qquad D = \frac{s-1}{1-s\lambda} - E, \qquad F = -\frac{q(1-\lambda) + \alpha^2/V_0}{q(1-s\lambda)}.$$

This E formula may be simplified by first substituting for λ from (3.6), then using $Q(V_0)/V_0 = 0$ to rewrite the term $(s-1)V_0$ in the denominator and finally cancelling a $(1 - \alpha/V_0)$ factor to give

$$E = \frac{2(1-\alpha)/\gamma}{s\alpha - 2(1-\alpha)/\gamma}.$$

Noting the η and q definitions in (3.4*b*) and (3.9*c*), the final forms of the constants are

$$E = \eta, \qquad D = \frac{s-1}{1-s\lambda} - \eta, \qquad F = \alpha - \frac{1-\lambda}{1-s\lambda}. \tag{4.2a,b,c}$$

Integration of (3.9*a*) and (4.1*a,b*) from the shock $\xi = 1$ provides

$$\frac{Z}{Z_s} = \left(\frac{V}{V_s}\right)^2 \left(\frac{\alpha - V_s}{\alpha - V}\right)^{\eta} \left(\frac{V+q}{V_s+q}\right)^{B}, \tag{4.3a}$$

$$\frac{G}{G_s} = \left(\frac{\alpha - V}{\alpha - V_s}\right)^{\eta} \left(\frac{V+q}{V_s+q}\right)^{D}, \tag{4.3b}$$

$$\xi = \left(\frac{V_s}{V}\right)^{\alpha} \left(\frac{V+q}{V_s+q}\right)^{F}, \tag{4.3c}$$

with V_s, Z_s and G_s given in (2.15). Non-dimensional forms of the flow variables follow from (2.14), (2.15) and (4.3) as

$$\frac{\rho}{\rho_s} = \left(\frac{\alpha - V}{\alpha - V_s}\right)^{\eta} \left(\frac{V+q}{V_s+q}\right)^{D}, \tag{4.4a}$$

$$\frac{u}{u_s} = \xi\frac{V}{V_s} = \left(\frac{V}{V_s}\right)^{1-\alpha} \left(\frac{V+q}{V_s+q}\right)^{F}, \tag{4.4b}$$

$$\frac{p}{p_s} = \xi^2\left(\frac{G}{G_s}\right)\left(\frac{Z}{Z_s}\right) = \left(\frac{V}{V_s}\right)^{2(1-\alpha)} \left(\frac{V+q}{V_s+q}\right)^{B+D+2F}. \tag{4.4c}$$

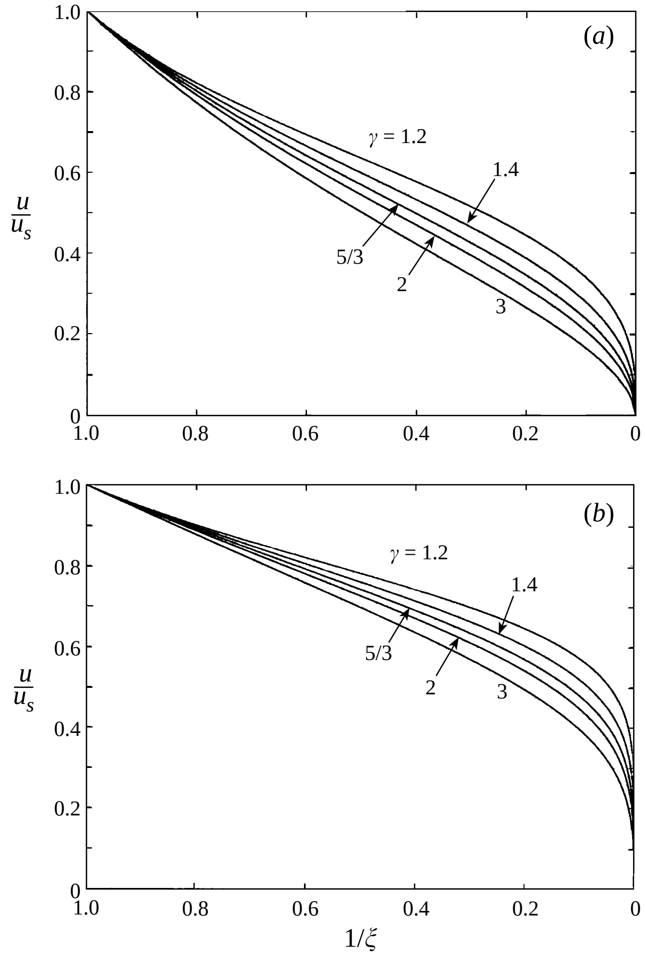


FIGURE 2. The fluid velocity, normalized with respect to its value at the shock, shown as a function of $1/\xi = R(t)/r$ for various γ. (*a*) Spherical shocks, (*b*) cylindrical shocks.

The monotonic decreasing variable u/u_s is displayed as a function of $1/\xi$ with the help of equation (4.3*c*) in figure 2(*a*) for spherical shocks and in figure 2(*b*) for cylindrical shocks for various values of γ.

The density increases monotonically behind the shock and has a non-zero limiting value far behind the shock. The inverse density variable ρ_s/ρ is displayed in figure 3(*a, b*) for spherical and cylindrical shocks for various values of γ.

The behaviour of the pressure is more complicated. In the Appendix it is demonstrated that the exact numerical solution of the differential equations leads to a monotonic decreasing pressure for both cylindrical and spherical shocks when $\gamma \geqslant 3$. Further it is shown that for $\gamma \leqslant 2$ the pressure has a single maximum behind the shock. The same behaviour is exhibited by the analytic solutions presented in this section. The location of the pressure maximum, when it exists, occurs at the value of V given by equation (4.4*c*) as

$$V = \frac{-2q(1-\alpha)}{B+D+2F+2(1-\alpha)}.$$

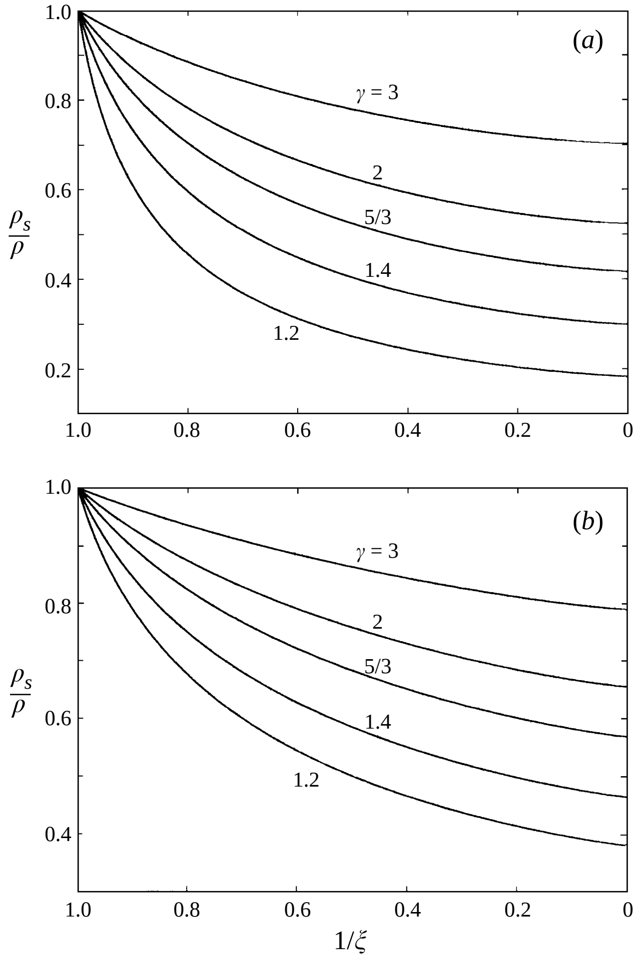
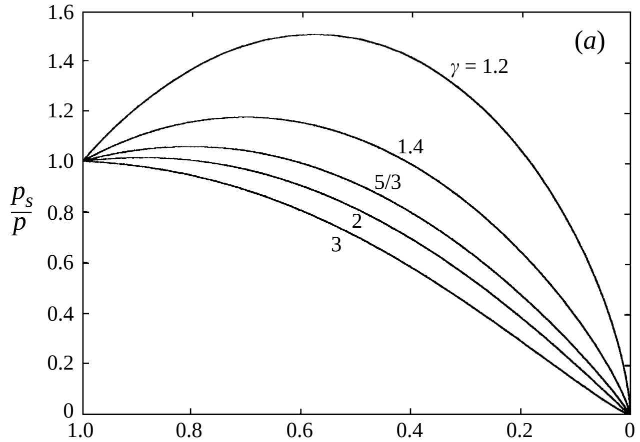


FIGURE 3. The inverse density, normalized with respect to its value at the shock, is shown as a function of $1/\xi$ for various γ. The density increases monotonically behind the shock and has a finite value far behind the shock where $1/\xi = 0$. (*a*) Spherical shocks, (*b*) cylindrical shocks.

After substitution for B, D, F and q this reduces to

$$V = \frac{2\alpha(1-\alpha)}{(s-1)\gamma V_0}. \tag{4.5}$$

Consideration of the product of the zeros of the quadratic $Q(V)$ given in (2.9) shows that the maximum value of the pressure occurs at the second zero of $Q(V)$, the first zero being V_0, the location of the singular point on $\varDelta = 0$. In the Appendix it is shown that this value of V also applies to the location of the pressure maximum in the full numerical solution to the problem, providing further strong evidence of the accuracy of the present analytic solution. The pressure maximum exists when the second zero lies in the flow range of V, namely $0, V_s$ and an inequality for this to occur is derived in the Appendix. For the approximate theory of this paper the pressure maximum occurs at the shock at $\gamma = 2.59$ for spherical shocks and $\gamma = 2.65$ for cylindrical shocks. The normalized pressure is displayed as a function of $1/\xi$ for



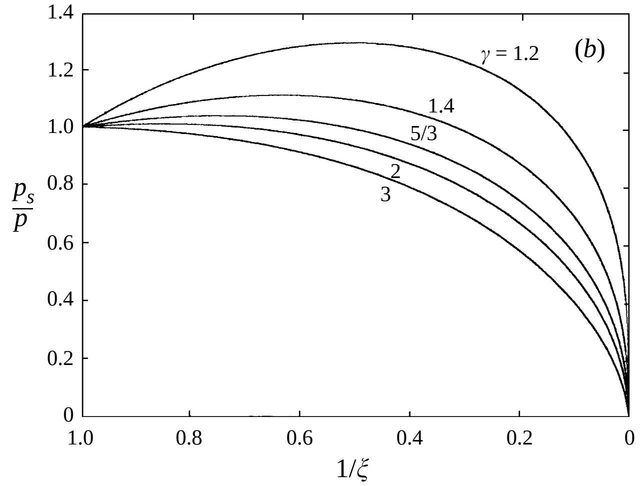


FIGURE 4. The pressure, normalized with respect to its value at the shock, shown as a function of $1/\xi$ for various γ. The pressure has a maximum in the flow behind the shock for values of $\gamma \leqslant 2$. (*a*) Spherical shocks, (*b*) cylindrical shocks.

spherical shocks in figure 4(*a*) and for cylindrical shocks in figure 4(*b*) for various γ.

The equations (2.6) do have one integral, known as the entropy integral. The integral obtained by adding an appropriate multiple of equations (2.6*a*) to (2.6*c*) is

$$\left(\frac{G}{G_s}\right)^{(1/s\alpha)-(\gamma-1)/(2(1-\alpha))}\left(\frac{\alpha-V}{\alpha-V_s}\right)^{1/s\alpha}\left(\frac{Z}{Z_s}\right)^{1/(2(1-\alpha))}\xi^{1/(\alpha(1-\alpha))}=1. \tag{4.6}$$

Although the analytic solution of this section does not satisfy any one of the equations of motion it does satisfy the entropy integral, providing further evidence of the accuracy of the analytic solution. To establish this result, equations (4.3) are substituted into the left-hand side of (4.6), providing powers of V, $\alpha - V$ and $V + q$. The V power consists of two equal and opposite terms and the other two powers may be shown to be zero after using the η definition in (3.4*b*).

5. The solution as $\gamma \to 1$: freely propagating shocks

The motion of the shock has an especially simple form when the ratio of specific heats tends to unity. This is because the region of disturbance which overtakes the shock during its passage to the origin shrinks to zero in this limit.

To demonstrate this result we first show that the boundary of the overtaking disturbance is both a $\xi =$ constant curve and a C_- characteristic. The C_- characteristics satisfy

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u - c, \tag{5.1}$$

which may be written as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\dot{R}\xi}{\alpha}(V + Z^{1/2}), \tag{5.2}$$

after first using (2.2*b,c*), with the square root for c appropriate to $t < 0$, and then using (2.5). The $\xi = r/R =$ constant lines all pass through the origin in the (r, t)-plane and their slope is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{R}\xi. \tag{5.3}$$

The singular point $V_0, Z_0 = (\alpha - V_0)^2$ on the solution curve of equation (2.11) will correspond to a particular value of $\xi = \xi_0$, obtained by integrating equation (2.12*b*) from V_s to V_0. As $V_0 < \alpha$ the relation $Z_0 = (\alpha - V_0)^2$ may be written $Z_0^{1/2} = \alpha - V_0$, showing that points in the flow which correspond to the singular point V_0, Z_0 have the property that the C_- characteristic (5.2) and the $\xi =$ constant curve (5.3) passing through them are the same. Thus the curve $\xi = \xi_0$, which passes through the origin, is a C_- characteristic and disturbances in the region $1 > \xi > \xi_0$ overtake the shock before it reaches the origin.

Equations (2.15*c,d*) show that $V_s \to \alpha$ and $Z_s \to 0$ as $\gamma \to 1$ and the $Z(V)$ integral curve remains close to the V-axis. The shock point V_s, Z_s and the singular point V_0, Z_0 coalesce at $\alpha, 0$ in this limit. When $V_0 \to \alpha$, equation (3.11) shows that $\alpha \to 1$, so that V_0 and hence $\xi_0 \to 1$ in the limit. Thus the region of overtaking disturbance shrinks to zero in this limit and the shock is said to be 'freely propagating'.

In an earlier paper (Chisnell 1957) the author considered 'freely propagating' cylindrical and spherical shock waves, based on a result of Chester (1954) for the motion of a shock in a channel containing a small area change. The simple approach of integrating Chester's result, and in consequence neglecting the disturbance which overtakes and modifies the path of the shock, led to an explicit relationship between the shock Mach number and the area of the channel. An alternative derivation of this result was given later by Whitham (1958) by an application of a characteristic rule and this result has been used for all γ in many subsequent papers on shock wave dynamics. For strong shocks the relationship becomes an explicit formula for the similarity exponent

$$\frac{(s-1)\alpha}{1-\alpha} = 1 + \frac{2}{\gamma} + \left(\frac{2\gamma}{\gamma - 1}\right)^{1/2}. \tag{5.4}$$

We now determine the asymptotic form of $Z(V)$ as $\gamma \to 1$ and the first three terms in a series for α in powers of $(\gamma - 1)^{1/2}$. An alternative derivation of (5.4) is provided by an extreme form of the differential equation defining the asymptotic form of $Z(V)$.

Equations (3.11) and (3.4b) show that $(\alpha - V_0)$ and η are both of the first order in $(1-\alpha)$

$$\alpha - V_0 \sim \frac{1-\alpha}{s-1}, \qquad \eta \sim \frac{2(1-\alpha)}{s}. \tag{5.5a,b}$$

At the singular point V_0, $Z_0 = (\alpha - V_0)^2$ and at the shock V_s equation (2.15) shows that

$$\alpha - V_s \sim \frac{\gamma - 1}{2}, \qquad Z_s \sim \frac{\gamma-1}{2}. \tag{5.6a,b}$$

To relate the small quantities $(1-\alpha)$ and $(\gamma - 1)$, we first consider the derivative of Z at the singular point V_0, which to the lowest order in $(1-\alpha)$ and $(\gamma-1)$ follows from (3.18) and (3.16) as

$$\frac{\mathrm{d}Z}{\mathrm{d}V}(V_0) \sim \frac{4s}{(s+1)}\frac{(1-\alpha)^2}{(s-1)^2} - \frac{(1-\alpha)(\gamma-1)}{(s+1)},$$

by noting that the term independent of ϕ in (3.18) is small. This suggests that the change in Z over the first-order range

$$V_s - V_0 \sim \frac{1-\alpha}{s-1} - \frac{\gamma-1}{2}$$

is of the third order in $(1-\alpha)$ and $(\gamma-1)$ and requires the first-order expression for Z_s and the second-order expression for Z_0 to balance, giving

$$\frac{\gamma-1}{2} \sim (\alpha - V_0)^2 \sim \frac{(1-\alpha)^2}{(s-1)^2}, \tag{5.7}$$

after using (5.5a). This is the required relationship between the small parameters $(1-\alpha)$ and $(\gamma-1)$ and provides a two-term expansion for α

$$\alpha = 1 - (s-1)\delta, \qquad \delta = \left(\frac{\gamma-1}{2}\right)^{1/2}. \tag{5.8a,b}$$

As Z increases from Z_0 to Z_s its slope increases by an order of magnitude to the value

$$\frac{\mathrm{d}Z}{\mathrm{d}V}(V_s) \sim \frac{2(1-\alpha)}{s} \tag{5.9}$$

due to the presence of the term $\eta/(\alpha - V)$ in equation (3.5). The resulting Z behaviour, $Z_s(\alpha - V_s)^\eta(\alpha - V)^{-\eta}$, has an increment from V_0 to V_s of

$$\frac{(1-\alpha)(\gamma-1)}{s}\ln\left(\frac{2(1-\alpha)}{(s-1)(\gamma-1)}\right).$$

Although this result confirms that the change in Z over (V_0, V_s) is of the third order in $(1-\alpha)$ it indicates that higher terms in the α-series (5.8) will contain ln terms in the coefficients. The presence of the ln terms in the coefficients was discovered in an earlier paper by the author (Chisnell 1987), using a matched asymptotic expansion approach which suggested a small power singularity as a possible cause of the ln coefficients.

To determine Z to the third order in $(1-\alpha)$, we note that both terms on the left-hand side of (3.5) are $O(1)$ at V_0 and although they are larger at V_s, their difference remains $O(1)$. We now show that use of just the first term of the Z-expansion, which is seen from (5.7) to be $(\alpha - V_0)^2$, enables the right-hand side of (3.5) to be determined to the same order over V_0, V_s. Sustituting $Z = (\alpha - V_0)^2$ into equation (2.8) gives Δ the second-order form

$$\Delta = (V - V_0)(V - V_0 - 2(\alpha - V_0)), \tag{5.10}$$

which combined with equation (3.7b) for $Q(V)$, enables the factor $(V - V_0)$ to be removed from the right-hand side of equation (3.5). Retaining only the first-order terms in $(1-\alpha)$, $(\alpha - V_0)$ and η in each of the two expressions in the resulting numerator and denominator leads to

$$\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}V} - \frac{\eta}{\alpha - V} = \frac{2(V - V_0 - 2(\alpha - V_0)) - \eta(s-1)V}{sV(V - V_0 - 2(\alpha - V_0)) + (\alpha - V)(s-1)V}.$$

The denominator of the right-hand side is $V(V - \alpha - s(\alpha - V_0))$ and to display the behaviour of Z near V_s, the second bracket of this denominator and the $(\alpha - V)$ on the left-hand side are written in terms of $(V - V_s)$. Expressing the right-hand side in partial fractions to $O(1)$ over V_0, V_s and using equations (5.5)–(5.8) to introduce the parameter δ gives

$$\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}V} - \frac{2(s-1)\delta}{s(\delta^2 + V_s - V)} \sim \frac{2}{V} + \frac{2(s-1)\delta}{s(V - V_s - s\delta)}. \tag{5.11}$$

Integration from V_s gives the asymptotic form of Z as $\gamma \to 1$ as

$$\frac{Z}{Z_s} \sim \left(\frac{V}{V_s}\right)^2 \left(\frac{1 + (V_s - V)/(s\delta)}{1 + (V_s - V)/\delta^2}\right)^{2(s-1)\delta/s}, \tag{5.12}$$

showing that in this limit Z/Z_s is determined directly in terms of γ and is not linked with the determination of α/V_0 and α as occurred in §3. The second term in the product on the right-hand side is just less than unity over $0, V_s$ and represents the modification needed to the trial function Z_T in this limit. The denominator in this term gives Z/Z_s the large slope $2(s-1)/(s\delta)$ at V_s, due to the singularity at $V = \alpha$, though Z itself has the small slope given in (5.9). As the expressions for $\varDelta$ given in (5.10) and (3.7a) have a difference of $O(\delta^3)$, the result (5.12) may alternatively be derived from equations (3.9).

The parameter α/V_0 is derived from (5.12) by substitution of $V = V_0$ and $Z = (\alpha - V_0)^2$. As $(V_s - V_0) \sim \delta$ and $Z_s/V_s^2 \sim \delta^2$, Z_0/V_0^2 follows as

$$\left(\frac{\alpha}{V_0} - 1\right)^2 = \delta^2 \left(\frac{(s+1)\delta}{s}\right)^{2(s-1)\delta/s}.$$

To determine the next term in the power series for α, expansion for α/V_0 gives

$$\frac{\alpha}{V_0} = 1 + \delta + \frac{(s-1)}{s}\delta^2 \ln\left(\frac{(s+1)\delta}{s}\right)\dots$$

and equation (3.11) then provides

$$\alpha = 1 - (s-1)\delta + (s-1)\delta^2\left(s + 2 - \frac{(s-1)}{s}\ln\left(\frac{(s+1)\delta}{s}\right)\right)\dots \tag{5.13}$$

as the first three terms in the power series for α. The series provides four-figure agreement with the numerical solution of (3.10) and (3.11) when $(\gamma - 1)$ is 10^{-4}.

The early result (5.4) can be derived from (5.11) by giving δ its limit value of zero. This extreme limiting form of Z is the trial function Z_T, given in (3.6), as $Z = (\alpha - V_0)^2$ at $V = V_0$. Use of this result for general γ and requiring Z to have the value Z_s at V_s, defined in equations (2.15), leads to

$$\left(\frac{\alpha}{V_0} - 1\right)^2 = \frac{\gamma(\gamma - 1)}{2}. \tag{5.14}$$

Substitution into (3.11) and noting that $\alpha > V_0$ provides

$$\frac{(s-1)\alpha}{1-\alpha} = \frac{1+\left(\frac{1}{2}\gamma(\gamma-1)\right)^{1/2}}{\left(\frac{1}{2}\gamma(\gamma-1)\right)^{1/2}} + \frac{2}{\gamma}\left(1+\left(\frac{1}{2}\gamma(\gamma-1)\right)^{1/2}\right),$$

which becomes (5.4) after some reduction.

We conclude the section by presenting the $\gamma \to 1$ asymptotic forms of the flow variables displayed in figures 2, 3 and 4. These may be obtained either by integrating the subsidiary equations (2.12) for G and ξ in a manner similar to the derivation of (5.12) from (3.5) or, because the Δ expressions in (5.10) and (3.7a) agree to $O(\delta^2)$, as the limiting forms of equations (4.3c) and (4.4). This leads to

$$\frac{1}{\xi} \sim \left(\frac{V}{V_s}\right)\left(1+\frac{V_s - V}{s\delta}\right)^{(s-1)\delta}, \tag{5.15a}$$

$$\frac{u}{u_s} \sim \left(\frac{V}{V_s}\right)^{(s-1)\delta}\left(1+\frac{V_s - V}{s\delta}\right)^{-(s-1)\delta}, \tag{5.15b}$$

$$\frac{\rho}{\rho_s} \sim \left(1+\frac{V_s - V}{\delta^2}\right)^{2(s-1)\delta/s}\left(1+\frac{V_s - V}{s\delta}\right)^{s-1}, \tag{5.15c}$$

$$\frac{p}{p_s} \sim \left(\frac{V}{V_s}\right)^{2(s-1)\delta}\left(1+\frac{V_s - V}{s\delta}\right)^{s-1}. \tag{5.15d}$$

The $\gamma \to 1$ limiting form of the normalized velocity displayed in figure 2 remains just under $u/u_s = 1$ and just to the left of $1/\xi = 0$ due to the presence of the small power of V in (5.15b). Near the shock the limiting form crosses the displayed profiles and reaches the shock with a slope of $(s-1)$, which is larger than the slopes of the displayed curves. The inverse density displayed in figure 3 has a slope at the shock which tends to infinity as $3(s-1)/\delta$ and the limiting value of the inverse density tends to zero as ξ becomes large. The pressure displayed in figure 4 has a slope at the shock of $-(s-1)/\delta$ and the maximum value of the pressure becomes infinitely large at a ξ value which behaves as $(2\delta)^{-1}$ as $\gamma \to 1$.

6. The solution as $\gamma \to \infty$

As a final check on the validity of the analytic solution of this paper, a comparison is made with the exact values of the similarity exponent for large values of γ. There is some confusion in the literature as Stanyukovich(1960) appears to have misquoted results for large γ. For cylindrical shocks he quotes $\alpha = 0.5$ which is the appropriate value for the line explosion problem. For spherical shocks he quotes $\alpha = 3/8$ which when substituted in $\alpha/(\alpha-1)$ provides an exponent of 0.6 which is close to the correct value and may have been his intended value. Mishkin & Fujimoto (1978b) quote values of α to three figures from a report by Lazarus & Richtmyer (1977). In this short section the exponent is calculated to more figures, confirming the work of Lazarus & Richtmyer and enabling a comparison to be made with the analytic description of this paper and the earlier shock wave dynamic result.

When γ becomes large, equations (2.15) show that the shock point V_s, Z_s tends to $0, 2\alpha^2$ and the solution curve remains close to the Z-axis. Introducing the scaled variable

$$v = \gamma V, \tag{6.1}$$

		Exact	This paper	CCW
(*a*)	α	0.588289	0.5945	0.547
	$\alpha/(1-\alpha)$	1.428890	1.4664	1.21
(*b*)	α	0.727048	0.7277	0.707
	$\alpha/(1-\alpha)$	2.663649	2.6725	2.41

TABLE 2. The values of the similarity exponents α and $\alpha/(1-\alpha)$ valid for infinite γ are given in the first column. These are obtained by finding a solution of equation (6.3) which passes through the singular point (6.4) and the shock point (6.5). The second column gives the corresponding results for the approximate solution of this paper obtained by solving the algebraic equations (6.7) and (6.9). The final column gives the corresponding shock wave dynamics results obtained from (6.10). (*a*) Spherical shocks, (*b*) cylindrical shock.

equations (2.8) and (2.9) show that

$$\Delta = -Z + \alpha^2 + O(1/\gamma), \qquad \gamma Q = 2\alpha(1-\alpha) + v(1 - s\alpha) + O(1/\gamma), \tag{6.2}$$

and equation (3.1) becomes to $O(1/\gamma)$

$$\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}v} = \frac{2Z - 2\alpha + v(s\alpha - 1)}{Z(sv - 2(1-\alpha)) - \alpha v}. \tag{6.3}$$

A solution of this limiting form of the equation is required which passes through the singular point on $\Delta = 0$ and the shock point v_s, Z_s. The singular point is given by (6.2) as

$$v_0 = \frac{2\alpha(1-\alpha)}{s\alpha - 1}, \qquad Z_0 = \alpha^2 \tag{6.4}$$

and the shock point follows from (2.15*c,d*) as

$$v_s = 2\alpha, \qquad Z_s = 2\alpha^2. \tag{6.5}$$

Linear analysis about this singular point provides the value of the derivative as

$$\frac{\mathrm{d}Z}{\mathrm{d}v} = \begin{cases} \frac{1}{4}\alpha(3\alpha - 1)(1 + (1 + 8/(1-\alpha))^{1/2}), & s = 3 \\ \frac{1}{4}\alpha(2\alpha - 1)(1 + (9 - 8\alpha)^{1/2})/(1-\alpha), & s = 2 \end{cases} \tag{6.6}$$

and enables the equation to be integrated numerically starting at the singular point. An iteration on α determines the solution which passes through the shock point. The similarity exponent α is given in the first column of table 2, which also lists $\alpha/(1-\alpha)$ for comparison with table 1.

The analytic solution of this paper can also be used for large γ. Introducing the scaled v variable into (3.11) provides

$$\frac{(s-1)\alpha}{1-\alpha} = 1 + \frac{2\alpha}{v_0}. \tag{6.7}$$

For large γ equation (3.4*b*) shows that $\eta \sim 0$ and equations (3.6), (3.9*b,c*) give

$$\gamma q \sim v_0/(s\alpha), \qquad B \sim -2(s-1)/s, \tag{6.8}$$

so that the limiting form of (3.10) is

$$\frac{\alpha^2}{v_0^2} = \frac{1}{2}\left(\frac{1 + s\alpha}{1 + 2\alpha(s\alpha/v_0)}\right)^{-2(s-1)/s}. \tag{6.9}$$

Eliminating α/v_0 from (6.7) and (6.9) provides an equation for α and the solutions are displayed in the second column of table 2. Thus even in the extreme situation of infinite ratio of specific heats the analytic solution of this paper is in reasonable agreement with the exact solution.

Finally the freely propagating, or CCW result given in (5.4), may be compared with the exact values. The limiting form of (5.4) for large γ is

$$\frac{(s-1)\alpha}{1-\alpha} = 1 + \sqrt{2} \tag{6.10}$$

and the results are shown in the third column of table 2. It is a tribute to this early result, which is based on $(\gamma - 1)$ being small, that it is still serviceable for infinite γ.

Appendix

In this Appendix we determine the condition for the pressure obtained in the exact numerical solution to have a maximum in the flow behind the shock and also the location of the maximum when it exists.

The sound speed definition shows that

$$\frac{1}{p}\frac{\partial p}{\partial r} = \frac{1}{c^2}\frac{\partial c^2}{\partial r} + \frac{1}{\rho}\frac{\partial \rho}{\partial r} \tag{A 1}$$

and equations (2.2*a,c*) provide

$$\frac{1}{p}\frac{\partial p}{\partial r} = \frac{2}{\xi R} + \frac{Z'}{RZ} + \frac{G'}{RG}. \tag{A 2}$$

Using (2.7*b,c*) for Z' and G' and (2.10*b,c*) to express Δ_2 and Δ_3 in terms of Δ and Q, it is found that the Δ terms in the numerator cancel and there results

$$\xi\frac{R}{p}\frac{\partial p}{\partial r} = -\frac{\gamma Q}{\Delta}. \tag{A 3}$$

This equation shows that the pressure can have turning points only at zeros of the quadratic $Q(V)$. One of these zeros corresponds to the singular point on the $Z(V)$ integral curve at its crossing of the parabola $\Delta = 0$. At this point $\partial p/\partial r$ is in general non-zero and equations (3.7) show that it is zero only in the special case of $Q(V)$ having two equal zeros. Thus the pressure can have at most one turning point and it will exist in the flow only if the second zero of $Q(V)$ lies in the range $0, V_s$. The quadratic always has one zero in the range $0, V_s$ and as $Q(0) = 2\alpha(1-\alpha)/\gamma$ is positive, a pressure turning point requires $Q(V_s)$ to be positive. The derivation of (3.11) from $Q(V_0) = 0$ shows that $Q(V)$ is a positive multiple of

$$-\frac{(s-1)\alpha}{1-\alpha} + \frac{2\alpha}{\gamma V} + \frac{\alpha}{\alpha - V} \tag{A 4}$$

and subsitution for V_s from (2.15*c*) shows that $Q(V_s)$ is positive provided

$$\frac{(s-1)\alpha}{1-\alpha} < \frac{\gamma+1}{\gamma} + \frac{\gamma+1}{\gamma-1} = \frac{(\gamma+1)(2\gamma-1)}{\gamma(\gamma-1)}. \tag{A 5}$$

Stanyukovich noted that spherical shocks had a pressure maximum for $\gamma = 1.4$ but not for $\gamma = 3$. Lazarus (1981) demonstrated that a pressure maximum exists for values of γ less than the critical γ at which $Q(V)$ has equal roots, namely 1.870 for spherical shocks and 1.909 for cylindrical shocks. The available numerical results show that the inequality (A 5) is satisfied, and a pressure maximum exists for $\gamma \leqslant 2$ for both cylindrical and spherical shocks. For $\gamma \geqslant 3$ the maximum does not exist and at some γ between 2 and 3 the maximum occurs at the shock and then the second zero of $Q(V)$ moves outside the flow range. For the approximate theory of this paper the inequality is satisfied for $\gamma \leqslant 2.59$ for spherical shocks and $\gamma \leqslant 2.65$ for cylindrical shocks.

When the pressure turning point exists it is a maximum. In §2 it was shown that $\Delta(V_s)$ is negative, so when $Q(V_s)$ is positive equation (A 3) shows that $\partial p/\partial r$ at the shock is positive, the pressure rising to a maximum before becoming small for large r.

Mishkin & Fujimoto (1978*b*) derived equation (A 5) as the condition for two possible stationary points in the pressure. Postulating that the pressure should have at most one turning point they determined the similarity exponent by requiring the two stationary points to coalesce. As explained above, one of the zeros of $Q(V)$ does not lead to a stationary point in the pressure and thus their reasoning is flawed as was pointed out by Lazarus (1980). Over the physical range of γ the zeros of $Q(V)$ are fairly close together and as a consequence a reasonable estimate of the similarity exponent is obtained in this manner. Yousaf (1986) demonstrated that the estimate of Mishkin & Fujimoto is equivalent to the first approximation of Stanyukovich in his iterative solution of the differential equations.

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